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On the geometry of the inhomogeneous Heisenberg ferromagnet: non-integrable case

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Abstract. New classes of exact solutions of the inhomogeneous Heisenberg ferromagnet equation are found by means of a geometrical technique. This technique consists of calculating one-parameter families of geodesics on surfaces in E^3. Our solutions correspond to geodesics on surfaces of rotation.

1. Introduction

In this paper we consider the classical version of the one-dimensional continuous system of spins with inhomogeneous nearest neighbour Heisenberg interaction. This model (called the inhomogeneous Heisenberg ferromagnet) is described by the following nonlinear system [1]

$$\boldsymbol{S}_{,2} = \boldsymbol{S} \wedge (f\boldsymbol{S}_{,1})_{,1} \tag{1a}$$

$$\boldsymbol{S} \cdot \boldsymbol{S} = 1 \tag{1b}$$

where $\boldsymbol{S} = \boldsymbol{S}(x^1, x^2)$ 'spin vector' is E^3-valued function of two real variables: x^1-space variable and x^2-time variable, the prime means differentiation, $\wedge(\cdot)$ denotes the skew (scalar) product in E^3, and, finally, $f = f(x^1, x^2)$ 'coupling function' is a given real function. There are some other physical systems which can be modelled by the system (1) [1]. The basic mathematical difficulties of the system (1) have been discussed in [2]. In the same paper some novel geometrical method to generate exact solutions to (1) has been set forth as well.

The present paper constitutes an essential extension of [2]. In particular, the algorithm of [2] is now applied to a class of surfaces of rotation. This algorithm proves to be efficient: we present several classes of exact solutions to (1). Some of them seem to be of physical value. In general, the paper is an attempt to give a possibly complete discussion of the subject outlined in [2].

2. Physics of the model

To begin with, let us discuss the underlying physics of the model (1). Consider the following mechanical system. Its configuration space is $S^2 \times S^2 \times \ldots \times S^2$ (N times): an instantaneous position of the system is given by $[\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_N]$. Thinking in terms of Hamiltonian formalism we can postulate the following general form of the equations of motion of our system

$$\frac{\mathrm{d}\boldsymbol{S}_j}{\mathrm{d}t} = \{\boldsymbol{S}_j, H\} \tag{2}$$

where H is Hamiltonian and $\{\,,\}$ is the Poisson bracket. Plainly, any choice of H and $\{\,,\}$ is assumed to respect the following constraints

$$\boldsymbol{S}_j(t) \cdot \boldsymbol{S}_j(t) = 1 \qquad (j = 1, 2, \ldots, N). \tag{3}$$

Our choice of H is as follows

$$H := \sum_{j=1}^{N-1} f_j(t) \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} \tag{4}$$

where $N-1$ functions f_i are assumed to be given.

Our choice of $\{\,,\}$ is defined by the following relations

$$\{S_j^a, S_k^b\} := \delta_{jk} \sum_{c=1}^{3} \varepsilon_{abc} S_j^c \tag{5}$$

where S_j^a are components of $\boldsymbol{S}_j$, δ_{jk} is Kronecker's delta symbol, and, finally, ε_{abc} is the three-dimensional totally antisymmetric symbol.

The choices (4) and (5), when substituted into (2), give the following equations of motion

$$\frac{\mathrm{d}\boldsymbol{S}_j}{\mathrm{d}t} = f_j(t) \boldsymbol{S}_j \wedge \boldsymbol{S}_{j+1} + f_{j-1}(t) \boldsymbol{S}_j \wedge \boldsymbol{S}_{j-1}. \tag{6}$$

Certainly, the dynamics (6) does not destroy (3).

Physically, one would say that the discussed model describes a system of N classical (no quantum-mechanical operators!) spins arranged along a line and subject to inhomogeneous nearest-neighbour Heisenberg interaction. The standard 'continuum limit' procedure performed on the model (6) leads to the model (1). Surely, the replacements $j \mapsto x^1$, $t \mapsto x^2$ are used. From now on we discuss this continuum version of our model, that is, the inhomogeneous Heisenberg ferromagnet (IHF) model exclusively. Intuitively, our model describes an open physical system. Indeed, for any solution $\boldsymbol{S}(x^1, x^2)$ of IHF model (1) the following identity holds

$$\mathscr{E}_{,2} + \mathscr{P}_{,1} = (\ln|f|)_{,2} \mathscr{E} \tag{7}$$

where

$$\mathscr{E} := \tfrac{1}{2} f \boldsymbol{S}_{,1} \cdot \boldsymbol{S}_{,1} \tag{8a}$$

$$\mathscr{P} := f^2 \boldsymbol{S} \cdot (\boldsymbol{S}_{,1} \wedge \boldsymbol{S}_{,11}). \tag{8b}$$

Equation (7) admits the following physical interpretation: $\mathscr{E}$ is the energy of our system, $\mathscr{P}$ is the energy flow density (almost momentum density) of our system, and, finally, $(\ln|f|)_{,2}\mathscr{E}$ is the energy source intensity. Certainly, the case $f_{,2}=0$ implies conservation of energy in our system: in general, the energy is not conserved (system is open).

The final observation is that the amount of emitted (absorbed) energy (per second) in our system is proportional to the energy density. A similar situation occurs in other physical systems, e.g. in systems of photons.

3. Non-integrability of the model

The IHF model (1) can be transformed into the following nonlinear system [1]

$$\mathrm{i}q_{,2}+(fq)_{,11}+2qR=0 \tag{9a}$$

$$R_{,1}=(f|q|^2)_{,1}+f_{,1}|q|^2 \tag{9b}$$

for two unknowns: complex function $q=q(x^1,x^2)$ and real function $R=R(x^1,x^2)$. The system (9) is called the inhomogeneous nonlinear Schrödinger (INS) equation. In fact, there exists (almost) a bijection between the smooth solution space of (1) and the smooth solution space of (9). This result, in turn, strongly suggests that the system (1) is integrable (in a sense of soliton theory) if the system (9) is integrable. Hence, to isolate the integrable cases of (1) one could work with, in a sense, the simpler system (9).

Indeed, for instance, Calogero and Degasperis proved the integrability of (9) for $f(x^1,x^2)=ax^1+b$ (a and b are real constants) [3]. This result can be easily extended to a, b being x^2 dependent. Are there any other integrable cases?

Several 'integrability tests' have been formulated in the literature [4-9]. To our best knowledge, the most general integrable case of (9) is given by $f(x^1,x^2)=a(x^2)x^1+b(x^2)$. In fact, in the forthcoming paper [10] we shall show that the application of two independent tests leads to the conclusion above.

This paper concerns a general (non-integrable) case.

4. Geometry of the model

One can associate some geometry of geodesics with the discussed model [2]. Consider an arbitrary smooth surface Σ in E^3. It is equipped (locally) in the so-called semi-geodesic coordinates, that is, orthogonal coordinates x^1 and x^2 meeting the requirement: $x^2=$ constant lines are geodesics on Σ. As a result, the metric on Σ assumes a form

$$\mathrm{I}=(\mathrm{d}x^1)^2+g(\mathrm{d}x^2)^2 \tag{10}$$

where $g=g(x^1,x^2)$ is some positive function. On the other hand, the second fundamental form

$$\mathrm{II}=b_{11}(\mathrm{d}x^1)^2+2b_{12}\,\mathrm{d}x^1\,\mathrm{d}x^2+b_{22}(\mathrm{d}x^2)^2 \tag{11}$$

where $b_{ij}=b_{ij}(x^1,x^2)$ are some functions, is still arbitrary.

As is well known from differential geometry, the coefficients of (10) and (11) necessarily satisfy the so-called Gauss–Mainardi–Codazzi equations. Simultaneously, the position vector $\boldsymbol{r}=\boldsymbol{r}(x^1,x^2)$ to Σ displays some 'kinematics', too: one can interpret $\boldsymbol{r}(x^1,x^2)$ as an evolution (in 'time x^2') of some string moving throughout E^3.

In this way we set up the geometric setting to unify a few physical models. Indeed, upon the identification

$$q = \tfrac{1}{2} b_{11} \exp(\mathrm{i}\psi) \tag{12a}$$

where ψ is subject to the constraint $\psi_{,1} = b_{12}/\sqrt{g}$, and

$$f = -\sqrt{g}/b_{11} \tag{12b}$$

$$R = \tfrac{1}{2}(\psi_{,2} - b_{22}/\sqrt{g}) \tag{12c}$$

one can show that the Gauss-Mainardi-Codazzi equations can be rewritten as the system (9). Interestingly enough, one of Mainardi-Codazzi equations admits a physical interpretation as the energy balance equation (7).

Simultaneously, the 'kinematics' of the position vector $\boldsymbol{r}(x^1, x^2)$ is given by

$$\boldsymbol{r}_{,2} = f \boldsymbol{r}_{,1} \wedge \boldsymbol{r}_{,11} \tag{13}$$

and, finally $\boldsymbol{S} := \boldsymbol{r}_{,1}$ solves the IHF model equations (1).

To sum up, our 'technology' to generate both: the 'coupling function' f and the corresponding solution $\boldsymbol{S}$ to (1) is as follows:

(1) Select a surface Σ in E^3.

(2) Compute one-parameter family of geodesics on Σ (in general a set of geodesics on a surface is '2-manifold').

(3) Compute the corresponding semi-geodesic coordinate system.

(4) Compute the corresponding functions g and b_{ij}, then (a) the resulting 'coupling function' is given by (12b), and (b) the resulting spin solution $\boldsymbol{S}$ to (1) (with f given by (a)) is $\boldsymbol{r}_{,1}$.

The most critical point of the algorithm is the second one: the success of the method depends on our abilities to integrate the equations of geodesics on Σ. From this point of view, the most tractable class of surfaces are surfaces of rotation. In a sense this paper constitutes a detailed discussion of the algorithm above applied to the surfaces of rotation.

5. Geodesics on surfaces of rotation

The surface of rotation, when referred to the standard parameters: s (arc length of generator) and φ (azimuthal angle), is given by

$$\boldsymbol{r}(s, \varphi) = [x(s),\ y(s) \cos \varphi,\ y(s) \sin \varphi] \tag{14}$$

where $x = x(s)$, $y = y(s)$ define the generator of the surface of rotation. The metric and the second fundamental form read

$$\mathrm{I} = \mathrm{d}s^2 + y^2\, \mathrm{d}\varphi^2 \tag{15a}$$

$$\mathrm{II} = (\ddot{x}\dot{y} - \dot{x}\ddot{y})\, \mathrm{d}s^2 + y\dot{x}\, \mathrm{d}\varphi^2 \tag{15b}$$

where the dot denotes differentiation with respect to s.

From (15a) we can compute the Christoffel symbols and form the corresponding equations for a geodesic $\boldsymbol{r}(s(\sigma), \varphi(\sigma))$ on the surface (14) (σ-arc length parameter along the geodesic). Most of the Christoffel symbols vanish and the resulting equations can be read as:

$$s_{,\sigma\sigma} - y\dot{y}(\varphi_{,\sigma})^2 = 0 \tag{16a}$$

$$\varphi_{,\sigma\sigma} + 2y^{-1}\dot{y}\varphi_{,\sigma} s_{,\sigma} = 0. \tag{16b}$$

(16) can be integrated once to yield

$$\varphi_{,\sigma} = Ay^{-2} \tag{17a}$$

$$(s_{,\sigma})^2 = 1 - A^2y^{-2} \tag{17b}$$

where A is a constant. (17a) is equivalent to the famous Clairaut relation [11]. From (17) we deduce

$$\sigma = \pm \int_{s_0(A)}^{s} y(y^2 - A^2)^{-1/2}\,\mathrm{d}s + \sigma_0 \tag{18a}$$

$$\varphi = \pm A \int_{s_0(A)}^{s} y^{-1}(y^2 - A^2)^{-1/2}\,\mathrm{d}s + B \tag{18b}$$

where σ_0, B are constants, and $s_0(A)$ is some function of A with values within the domain of $y(s)$. There is no loss in generality in putting $\sigma_0 = 0$. In this way we have obtained a two (A, B) parameter family of geodesics on the surface of rotation (14).

We rewrite (18) as follows

$$\sigma = \sigma(s; A) \tag{19a}$$

$$\varphi = \varphi(s; A, B). \tag{19b}$$

Surely, we assume (19a) can be inverted

$$s = s(\sigma; A) \tag{20a}$$

and, as a result,

$$\varphi = \varphi(s(\sigma; A); A, B) := \varphi(\sigma; A, B). \tag{20b}$$

(20) is a (formal) general integral of the equations of geodesics on the surface of rotation (14).

Before entering into the further steps of our algorithm of section 4, we shall present two technical lemmas which enable one to simplify further calculations.

Lemma 1. Functions (19) satisfy the constraint

$$(\sigma_{,A} - A\varphi_{,A})_{,s} = 0 \tag{21}$$

in other words we can always put $\sigma_{,A} - A\varphi_{,A} := F(A)$.

On use of (17), lemma 1 can be proved directly. It is less trivial to show that the following sufficient conditions (22a) or (22b) lead to $F(A) = 0$.

Lemma 2. Suppose the functions (20) satisfy either

$$s(0; A) = 0 \qquad \text{and} \qquad \varphi(0; A, B) = 0 \tag{22a}$$

or

$$\varphi(0; A, B) = 0 \qquad \text{and} \qquad \varphi_{,\sigma}(0; A, B) = A^{-1} \tag{22b}$$

then the functions (19) are subject to the constraint

$$\sigma_{,A} = A\varphi_{,A}. \tag{23}$$

We point out that in all the cases discussed in subsequent sections one of the conditions (22) is always satisfied, and, hence (23) holds, too.

Let us choose a one-parameter family of geodesics: $A = A(\tau)$ and $B = B(\tau)$ in such a way that

$$(\sigma, \tau) \mapsto (s, \varphi) \tag{24}$$

is a local diffeomorphism. To be more explicit, on the use of (19*b*) and (20*a*) we have

$$s = s(\sigma; A(\tau)) \tag{25a}$$

and

$$\varphi = \varphi(s(\sigma; A(\tau)); A(\tau), B(\tau)). \tag{25b}$$

Surely, new local coordinates (σ, τ) are characterized by properties: $\tau = $ constant line is a geodesic, and σ is the arc length parameter along the geodesic. It is almost a semi-geodesic coordinate system: still it is not an orthogonal one. Indeed, one can show that the metric (15*a*) in (σ, τ) coordinates assumes the form

$$I = (\mathrm{d}\sigma + AB'\,\mathrm{d}\tau)^2 + (y^2 - A^2)(B' + A'A^{-1}\sigma_{,A})^2\,\mathrm{d}\tau^2 \tag{26}$$

where prime denotes $\mathrm{d}/\mathrm{d}\tau$, $y = y(s(\sigma; A(\tau)))$, and $\sigma_{,A} = (\partial/\partial A)\sigma(s; A)$ is evaluated at $s = s(\sigma; A(\tau))$ and $A = A(\tau)$. It is worth remarking that in establishing (26) one uses (23).

Surely, new coordinates

$$x^1 = \sigma + \int^{\tau} A(t)B'(t)\,\mathrm{d}t \tag{27a}$$

$$x^2 = \tau \tag{27b}$$

are semi-geodesic:

$$I = (\mathrm{d}x^1)^2 + (y^2 - A^2)(B' + A'A^{-1}\sigma_{,A})^2(\mathrm{d}x^2)^2 \tag{28}$$

where all involved functions are to depend on x^1 and x^2.

For completeness we shall list now all the coefficients of the second fundamental form (11) in the discussed case.

$$b_{11} = (\dot{x}^2A^2 - (y^2 - A^2)y\ddot{y})/y^3\dot{x} \tag{29a}$$

$$b_{12} = g^{1/2}A(y^2 - a^2)^{1/2}(\dot{x}^2 + y\ddot{y})/y^3\dot{x} \tag{29b}$$

$$b_{22} = g(\dot{x}^2(y^2 - A^2) - A^2y\ddot{y})/y^3\dot{x} \tag{29c}$$

where

$$\dot{x} = \pm(1 - \dot{y}^2)^{1/2} \tag{30a}$$

and g, see (10) and (28), is given by

$$g = (y^2 - A^2)(B' + A'A^{-1}\sigma_{,A})^2. \tag{30b}$$

Once more we point out that in (29) and (30) all the involved functions should depend explicitly on x^1 and x^2. This condition, in turn, requires the explicit knowledge of (20*a*). As a rule, this is the most critical point in the discussed setting.

To sum up, starting from the general surface of rotational (14), we have just described a method to compute a generic semi-geodesic coordinate system upon it. This, in turn, means that we are in a position to start up calculations of exact solutions to IHF model equations (1) described in section 4. The output results are described below.

6. New classes of solutions to IHF model equations

We are now able to establish the following general result.

Theorem. On use of (29*a*) and (30)

$$f = -\sqrt{g}/b_{11} \tag{31}$$

is computed as a function of x^1, x^2, then necessarily

$$\boldsymbol{S} = \left[\dot{x}\sqrt{1-\left(\frac{A}{y}\right)^2},\ \dot{y}\sqrt{1-\left(\frac{A}{y}\right)^2}\cos\varphi - \frac{A}{y}\sin\varphi,\ \dot{y}\sqrt{1-\left(\frac{A}{y}\right)^2}\sin\varphi + \frac{A}{y}\cos\varphi\right] \tag{32}$$

as a function of x^1, x^2 solves the IHF model equations (1) with f given by (31).

Several remarks are in order.

(i) Formula (32) represents a pretty large class of solutions to our model equations. These are parametrized by arbitrary functions: $y = y(s)$, $A = A(x^2)$ and $B = B(x^2)$.

(ii) In (32) $\dot{x}$ is given by (30*a*).

(iii) In general, formula (32) is implicit. It is explicit if, and only if, all the functions of (32) depend explicitly on x^1 and x^2. This, in turn, means we have to know (20*a*) explicitly, as well.

(iv) The sign ambiguity of square roots in (31) and (32) is resolved by the obvious requirement that the RHS of (31) and (32) should be smooth functions: the square root always changes a sign at a 'point of return', e.g. at point of root vanishing.

(v) In any case one should respect the obvious conditions: $y^2 \geqslant A^2$ and $\dot{y}^2 \leqslant 1$.

Fortunately, the condition (iii) can be satisfied in many cases. Some of them are listed in two subsequent tables. Before entering into some details of the tables, we proceed to the discussion of some general feature of the class (32) which, mathematically, is coded in (27), whereas physically, represents a phenomenon of wave propagation throughout an inhomogeneous medium.

Let us look more closely into the structure of (32). (32) is built from the following ingredients: y, $\dot{x}$, $\dot{y}$, φ and A. A depends upon x^2 in a trivial way: $A = A(x^2)$, whereas the dependence of y, $\dot{x}$, $\dot{y}$ and φ and x^1 and x^2 is fairly complicated. For instance,

$$y = y(s) = y[s(\sigma; A)] = y\left[s\left(x^1 - \int^{x^2} A(t)B'(t)\,\mathrm{d}t;\ A(x^2)\right)\right]. \tag{33}$$

Similar expressions can be written down for the remainder: $\dot{x}$, $\dot{y}$ and φ.

In our opinion, it is a pretty remarkable result that all this geodesic setting leads to the 'physical' argument

$$x^1 - \int^{x^2} A(t)B'(t)\,\mathrm{d}t \tag{34}$$

in (32). Indeed, the presence of (34) in (32) means that our generic solution (32) describes a modulated (by $A(x^2)$) travelling wave with variable (due to inhomogeneity!) phase velocity $\vartheta_{\mathrm{ph}}(x^2) = A(x^2)B'(x^2)$.

In particular, the choice

$$A(x^2) = \text{constant} \qquad \text{and} \qquad B(x^2) = x^2 \tag{35}$$

kills A-modulations, and the resulting wave is propagated with a constant phase velocity. Geometrically, this choice means that we select arbitrarily some geodesic on the surface of rotation and, subsequently, we rotate it steadily around x-axis. The simplest geodesic on the surface of rotation is the meridian of the surface ($A(x^2)=0$). In the discussed case ($A(x^2)=0$, $B(x^2)=x^2$) the following simplifications occur

$$s(x^1, x^2)=x^1 \qquad \text{and} \qquad \varphi(x^1, x^2)=x^2 \tag{36}$$

and, correspondingly,

$$f(x^1, x^2)=\frac{y(x^1)\sqrt{1-[y_{,1}(x^1)]^2}}{y_{,11}(x^1)} \tag{37}$$

is a function of x^1 exclusively, whereas (32) assumes a simple form

$$S(x^1, x^2)=[\sqrt{1-[y_{,1}(x^1)]^2},\ y_{,1}(x^1)\cos x^2,\ y_{,1}(x^1)\sin x^2]. \tag{38}$$

In (37) and (38) $y(x^1)$ is an arbitrary function fulfilling the constraint $y_{,1}^2 \leqslant 1$. We recall that thanks to (30a) and (36) $y=y(x^1)$ defines a generator of the surface of rotation, and, as a result, an instantaneous spin configuration is identical with the distribution of unit tangent vectors to the meridian ('spin on meridians').

On using (37) and (38), we can immediately produce a good deal of solutions to (1). The problem is to isolate some physically interesting cases. Several such solutions are collected in table 1. Notice that the coupling function of the case 9 is positive (we assume the square root changes its sign at the points of square root vanishing), while in other cases we can easily make f positive by an appropriate choice of the p-parameter value. Basically, our model (1) describes two different cases: the ferromagnetic one (f is positive), and the antiferromagnetic one (f is negative). Thus, table 1 concerns the ferromagnetic case.

The functions f listed in table 1 are regular for any $x^1 \in R$ and bounded (the only exception is the case 10). The coupling functions f of cases 2–7 and the corresponding solutions S are periodic. The examples 8 and 9 deserve special attention. The coupling function of 9 has a 'bell-like' shape and tends to 0 for $x^1 \to \pm\infty$. In the case 8 f tends to a constant (γ^2) for $x^1 \to \pm\infty$ and has one minimum, for $x^1=0$. In both of these cases $S \to [1, 0, 0]$ for $x^1 \to \pm\infty$.

To find any one-parameter family of geodesics different from meridians is not an easy task: the integrals (18) are not trivial for $A \neq 0$. Nevertheless in several cases, listed in table 2, we found explicit expressions for $s=s(\sigma; A)$ and $\varphi=\varphi(\sigma; A, B)$ in terms of elementary functions. To obtain the coupling function f and the solution S, one has to substitute $s=s(\sigma; \alpha)$ and $\varphi=\varphi(\sigma; \alpha, B)$, given in the table 2, into (29) and (30) and to use our theorem. The obtained solutions are usually much more complicated than those of the table 1, especially where their dependence on x^2 is concerned. The coupling functions are not positive, which may cause difficulties in the physical interpretation of these results. The cases 1, 2 and 3 of table 2 were considered in [2]. In case 1 (cylinder) the coupling function

$$f=-(p\alpha'/\sin^2\alpha)\sigma-(p^2B'\cos\alpha/\sin^2\alpha) \tag{39a}$$

is a function linear in x^1. Thus the solution S obtained by our method

$$S=[\cos\alpha,\ -\sin\alpha\sin(B+(\sigma/p)\sin\alpha),\ \sin\alpha\cos(B+(\sigma/p)\sin\alpha)] \tag{39b}$$

is a solution to the integrable sub-case of (1). Also, the choice (35) corresponds to the integrable sub-case $f=$ constant.

In case 2 (sphere) the coupling function

$$f=-\alpha'\sin\sigma-B'\cos\alpha\cos\sigma \tag{40a}$$

Table 1. 'Spins on meridians'. Notation: k-modulus of elliptic functions (if necessary), $\xi := x^1/\gamma$, δ, γ, p are constant parameters.

Case	Coupling function $f=$		Corresponding solution to (1) $S=(\dot x, \dot y\cos x^2, \dot y \sin x^2)$: $\dot x =$	$\dot y =$
1	γ^2	$k=1/\sqrt{2}$	$-\mathrm{cn}^2\,\xi$	$-\sqrt{2}\,\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi$
2	$p+\gamma^2\sin\xi$	(torus)	$-\sin\xi$	$\cos\xi$
3	$[p-k\cdot\sin^{-1}(k\,\mathrm{sn}\,\xi)]/\mathrm{dn}\,\xi$	$\xi=x^1/k$	$\mathrm{sn}\,\xi$	$\mathrm{cn}\,\xi$
4	$(\gamma^2/k^2)+p/\mathrm{dn}\,\xi$		$2\,\mathrm{sn}^2\,\xi-1$	$-2\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi$
5	$\frac{1}{2}\gamma\delta[p+\log(\delta\,\mathrm{dn}\,\xi)]/\cosh[\log(\delta\,\mathrm{dn}\,\xi)]$	$\gamma=1-\sqrt{1-k^2}$, $\delta=(1-k^2)^{-1/4}$	$\dfrac{k^2\,\mathrm{sn}^2\,\xi-\gamma}{\gamma\,\mathrm{dn}\,\xi}$	$\dfrac{-k^2\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi}{\gamma\,\mathrm{dn}\,\xi}$
6	$(1+\gamma^2\sin^2\xi)\left[p+\dfrac{\gamma}{\sqrt{1+\gamma^2}}\sinh^{-1}(\gamma\sin\xi)\right]$		$-\dfrac{\sqrt{1+\gamma^2}\sin\xi}{\sqrt{1+\gamma^2\sin^2\xi}}$	$\dfrac{\cos\xi}{\sqrt{1+\gamma^2\sin^2\xi}}$
7	$(1-\gamma^2\cos^2\xi)\left[p+\dfrac{\gamma}{\sqrt{1-\gamma^2}}\cos^{-1}(\gamma\cos\xi)\right]$		$\dfrac{\sqrt{1-\gamma^2}\cos\xi}{\sqrt{1-\gamma^2\cos^2\xi}}$	$\dfrac{\sin\xi}{\sqrt{1-\gamma^2\cos^2\xi}}$
8	$\dfrac{(4\gamma^2\sinh^2\xi+\delta^2)}{8\delta\cosh\xi}\log\left(\dfrac{\cosh\xi+\delta}{\cosh\xi-\delta}\right)$	$\delta=2\gamma/\sqrt{1+4\gamma^2}$	$\dfrac{4\gamma^2\sinh^2\xi-\delta^2}{4\gamma^2\sinh^2\xi+\delta^2}$	$\dfrac{-4\gamma\delta\sinh\xi}{4\gamma^2\sinh^2\xi+\delta^2}$
9	$\gamma^2\sqrt{1-\xi^2\exp(1-\xi^2)}/(\xi^2-1)$		$\sqrt{1-\xi^2\exp(1-\xi^2)}$	$-\xi\exp[\frac{1}{2}(1-\xi^2)]$
10	$\gamma^2(1+\xi^2)^{3/2}$	(catenoid)	$1/\sqrt{1+\xi^2}$	$\xi/\sqrt{1+\xi^2}$

Table 2. 'Solutions given in terms of elementary functions'. According to our theorem S given by (32) solves the IHF equations (1) with f given by $f = y^3\dot{x}\sqrt{y^2-A^2}(B'+A'A^{-1}\sigma_{,A})/((y^2-A^2)y\ddot{y}-\dot{x}^2A^2)$. Below we list several functions $y=y(s)$ for which $s=s(\sigma;A)$ and $\varphi=\varphi(\sigma;A,B)$ can be expressed by elementary functions.

No.	$y=y(s)$ p, γ – constant	$A=A(\alpha)$ $\alpha=\alpha(x^2)$	$s=s(\sigma;\alpha)$ $\sigma=x^1-\int^{x^2}A(t)B'(t)\,dt$	$\varphi-B=\varphi(\sigma;\alpha,B)-B$ $B=B(x^2)$
1	p (cylinder)	$p\sin\alpha$	$s=\sigma\cos\alpha$	$(\sigma/p)\sin\alpha$
2	$s\sin\gamma$ (cone)	$\alpha\sin\gamma$	$s^2=\sigma^2+\alpha^2$	$\arctan(\sigma/\alpha)/\sin\gamma$
3	$\cos s$ (sphere)	$\sin\alpha$	$\sin s=\sin\sigma-\cos\alpha$	$\arctan(\sin\alpha\tan\sigma)$
4	$\sin\gamma\cos s$	$\sin\gamma\sin\alpha$	$\sin s=\sin\sigma\cos\alpha$	$\dfrac{\arctan(\sin\alpha\tan\sigma)}{\sin\gamma}$
5	$\tanh s/\cosh\gamma$	$\tanh\alpha/\cosh\gamma$	$\cosh s=\cosh\alpha\cosh\xi$ $\quad\xi=\sigma/\cosh\alpha$	$\cosh\gamma\,[\sigma\tanh\alpha+\arctan(\tanh\xi/\sinh\alpha)]$
6	$1/\sqrt{1+s^2}$	$\sin\alpha$	$s=\cot\alpha\sin\xi$ $\quad\xi=\sigma\sin\alpha$	$\xi+\frac{1}{4}(2\xi-\sin 2\xi)\cot^2\alpha$
7	$\dfrac{sp\sin\gamma}{\sqrt{s^2+p^2}}$	$p\sin\gamma\sin\alpha$	$s^2=\sigma^2\cos^2\alpha+p^2\tan^2\alpha$	$\dfrac{\sigma\sin\alpha+p\arctan(\sigma\cos^2\alpha/p\sin\alpha)}{p\sin\gamma}$
8	$\dfrac{p\sin\gamma\cos s}{\sqrt{p^2\sin^2 s+\cos^2 s}}$	$p\sin\gamma\sin\alpha$	$\sin s=\dfrac{1}{\lambda}\cos\alpha\sin(\lambda\sigma)$ $\quad\lambda=\sqrt{p^2\sin^2\alpha+\cos^2\alpha}$	$\dfrac{(1-p^2)\sigma\sin\alpha+p\arctan(p\sin\alpha\tan\xi/\lambda)}{p\sin\gamma}$
9	$\dfrac{4\sin\gamma}{p(1+e^{ps})}$	$\dfrac{4\sin\gamma\sin\alpha}{p}$	$e^{-ps}=\dfrac{\sin\alpha}{\cos^2\alpha}(\cosh\xi+\sin a)$ $\quad\xi=\sigma p\cos\alpha$	An elementary function but very complicated
10	$\dfrac{sp^2\sin\gamma}{p^2+s^2}$	$\frac{1}{2}p\sin\gamma\sin\alpha$	$s^2=\dfrac{p^2(1+\cos^2\alpha+2\sin\xi\cos\alpha)}{\sin^2\alpha}$ $\quad\xi=\dfrac{1}{p}\sigma\sin\alpha$	An elementary function but very complicated

is periodic and bounded, and the corresponding solution is

$$\boldsymbol{S}=[\cos\alpha\cos\sigma,\ -\cos B\sin\sigma-\sin\alpha\sin B\cos\sigma,\ \cos B\cos\sigma\sin\alpha-\sin B\sin\sigma] \tag{40b}$$

The coupling function corresponding to case 3 (cone) is unbounded:

$$f=\alpha^{-2}\tan\gamma(\alpha'-B'\sigma\sin\gamma)(\sigma^2+\alpha^2)^{3/2} \tag{41a}$$

and the corresponding solution still has a rather simple form

$$\boldsymbol{S}=(\sigma^2+\alpha^2)^{-1/2}[\sigma\cos\gamma,\ \sigma\sin\gamma\cos\varphi-\alpha\sin\varphi,\ \sigma\sin\gamma\sin\varphi+\alpha\cos\varphi]. \tag{41b}$$

We collect the formulae (39)-(41) in order to correct some misprints in the paper [2].

7. Concluding remarks: physics

Interestingly enough, the important physical quantities, energy density and energy flow density (8), can be expressed in a purely geometrical way:

$$\mathscr{E}=-\tfrac{1}{2}\sqrt{g}\,b_{11} \tag{42a}$$

$$\mathscr{P}=\sqrt{g}\,b_{12} \tag{42b}$$

where g and b_{ij} are defined in (10) and (11) correspondingly. Thus, on using (29) and (30) we are in a position to compute $\mathscr{E}$ and $\mathscr{P}$ in the discussed case. For instance, in the case of 'spins on meridians' ($A(x^2)=0$) we have

$$\mathscr{P}=0 \tag{43}$$

(stationary state). Also, for a general solution associated with the sphere (40) equation (43) holds.

Physically interesting solutions are characterized, for example, by bounded energy density (42a). Every solution from table 1 has this property, whereas some of the solutions gathered in table 2 have an unbounded energy density. Nevertheless, by an appropriate choice of parameters (usually $A(x^2)=\text{constant}$) we can make the energy density bounded.

Generic coupling functions f of table 1 are always positive and bounded. This implies that solutions of this table represent states of the ferromagnet.

The case of f being a negative function (antiferromagnet models) deserves a special attention. First of all, we notice that equations (1) admit the following discrete symmetry

$$\boldsymbol{S}\mapsto-\boldsymbol{S} \qquad \text{and} \qquad f\mapsto -f$$

which enables one to generate antiferromagnet solutions from ferromagnet solutions. On the other hand, the continuum limit of the antiferromagnet model, in contrast to the ferromagnet one, seems to be of a limited value. Presumably, within the continuum approximation, one can study states far from the stable equilibrium only. Fortunately, our solutions seem to satisfy this requirement.

Finally, we point out that a general coupling function of table 2 is of a variable sign. The same applies to the celebrated case of the integrable model $f(x^1, x^2) = a(x^2)x^1 + b(x^2)$. Can one attribute any physical meaning to the cases of this kind?

8. Concluding remarks: geometry

In this paper we confined ourselves almost completely to the discussion of the case of surfaces of rotation. The subject of the geodesics on surfaces of rotation seems to be a well established piece of classical differential geometry. Interestingly enough, some of our computational observations discussed in the paper (e.g. two lemmas) seem to be novel in this context.

This 'geodesic' method offers some promising possibilities to compute new classes of the exact solutions to IHF model equations (1). One open possibility is related to a fascinating and still fresh subject of the geodesics on quadrics [12]. To point out a more exotic case which, perhaps, deserves a special study, we recall that there exist the so-called Zoll surfaces. A surface is called the Zoll surface if all its geodesics are closed [13].

Finally, we point out that our approach admits a nice geometric formulation. It turns out that the totality of all (complete) geodesics of some Riemannian manifolds M can be also organized as a new manifold $G(M)$ [14]. The essence of our method can be described as follows: we are interested in drawing curves on $G(M)$ where M is a surface in E^3.

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